

LIFTING GALOIS REPRESENTATIONS TO RAMIFIED COEFFICIENT FIELDS

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ABSTRACT. Let $p > 5$ be a prime integer and K/\mathbb{Q}_p a finite ramified extension with ring of integers \mathcal{O} and uniformizer π . Let $n > 1$ be a positive integer and $\rho_n : G_{\mathbb{Q}} \rightarrow \mathrm{GL}_2(\mathcal{O}/\pi^n)$ be a continuous Galois representation. In this article we prove that under some technical hypotheses the representation ρ_n can be lifted to a representation $\rho : G_{\mathbb{Q}} \rightarrow \mathrm{GL}_2(\mathcal{O})$. Furthermore, we can pick the lift restriction to inertia at any finite set of primes (at the cost of allowing some extra ramification) and get a deformation problem whose universal ring is isomorphic to $W(\mathbb{F})[[X]]$. The lifts constructed are “nearly ordinary” (not necessarily Hodge-Tate) but we can prove the existence of ordinary modular points (up to twist).

1. INTRODUCTION

The present article is a continuation of the work done in [CP14], where we constructed, for a finite field \mathbb{F} , lifts of representations $\rho_n : G_{\mathbb{Q}} \rightarrow \mathrm{GL}_2(W(\mathbb{F})/p^n)$ to $\mathrm{GL}_2(W(\mathbb{F}))$. Here we prove how to extend the results to finite ramified extensions K/\mathbb{Q}_p of ramification degree e .

The method used in [CP14] followed the ideas of [Ram99] and [Ram02], adapted to the modulo p^n setting. As noticed in [CP14] (see the remark before Proposition 5.9) these methods do not generalize to representations $\rho_n : G_{\mathbb{Q}} \rightarrow \mathrm{GL}_2(\mathcal{O}/\pi^n)$ where \mathcal{O} is the ring of integers of K .

The obstacle is that the modulo π^2 reduction of ρ_n (which we denote ρ_2) fixes the same field extension of \mathbb{Q} as an element $f \in H^1(G_{\mathbb{Q}}, \mathrm{Ad}^0 \bar{\rho})$ (where $\mathrm{Ad}^0 \bar{\rho}$ is the adjoint representation of the reduction mod π of ρ_n). This implies that whenever we define a local condition for deformations containing ρ_n at a prime v it automatically contains ρ_2 and therefore f lies inside its tangent space. On this way, no matter which set of primes M we choose, the morphism of Theorem A of [?]

$$(1) \quad H^1(G_M, \mathrm{Ad}^0 \bar{\rho}) \rightarrow \bigoplus_{v \in M} H^1(G_v, \mathrm{Ad}^0 \bar{\rho})/N_v$$

will always have non-trivial kernel and therefore will never be an isomorphism (as required in [CP14]).

The key innovation of this work is to relax local conditions so that the morphism (1) is no longer an isomorphism, but a surjective map with one dimensional kernel. This will be enough for the lifting purpose, since it allows to find global elements that make the required the local adjustments at each step. We cannot relax conditions at primes $v \neq p$, since the local deformation ring of $\bar{\rho}|_{G_v}$ should not have a smooth quotient of dimension bigger than $\dim H^1(G_v, \mathrm{Ad}^0 \bar{\rho}) - \dim H^2(G_v, \mathrm{Ad}^0 \bar{\rho})$. Therefore, we need to impose a different local condition at the prime p . The condition we impose is the same as in [CM09].

Definition. We say that a deformation is “nearly ordinary” if its restriction to the inertia subgroup is upper-triangular and its semisimplification is not scalar, i.e. if

$$\rho|_{I_p} = \begin{pmatrix} \psi_1 & * \\ 0 & \psi_2 \end{pmatrix}.$$

with $\psi_1 \neq \psi_2$.

2010 *Mathematics Subject Classification.* 11F33; 11F80.

Key words and phrases. Galois Representations; Modular forms.

The author was partially supported by a CONICET doctoral fellowship.

Using this local condition at p , we are able to derive a slightly weaker version of the following theorem (see Theorem 4.1 for the precise statement), which is one of the main results of this work.

Theorem. *Let $\rho_n : G_{\mathbb{Q}} \rightarrow \mathrm{GL}_2(\mathcal{O}/\pi^n)$ be a continuous representation which is odd and nearly ordinary at p . Assume that $\mathrm{Im}(\rho_n)$ contains $\mathrm{SL}_2(\mathcal{O}/p)$ if $n \geq e$ and $\mathrm{SL}_2(\mathcal{O}/\pi^n)$ otherwise. Let P be a set of primes of \mathbb{Q} containing the ramification set of ρ_n . For each $v \in P \setminus \{p\}$ fix a local deformation $\rho_v : G_v \rightarrow \mathrm{GL}_2(\mathcal{O})$ that lifts $\rho_n|_{G_v}$. Then there exists a continuous representation $\rho : G_{\mathbb{Q}} \rightarrow \mathrm{GL}_2(\mathcal{O})$ and a finite set of primes R such that:*

- ρ lifts ρ_n , i.e. $\rho \equiv \rho_n \pmod{\pi^n}$.
- ρ is unramified outside $P \cup R$.
- For every $v \in P$, $\rho|_{I_v} \simeq \rho_v|_{I_v}$.
- ρ is nearly ordinary at p .
- All the primes of R , except possibly one, are not congruent to 1 modulo p .

In fact, the method provides us not only a lift of ρ_n to \mathcal{O} but a family of lifts to characteristic 0 rings, parametrized by a lift to the coefficient ring $W(\mathbb{F})[[X]]$ (see Theorem 4.13). The downside is that this family of representations is not ordinary but nearly ordinary, which implies that most points are not Hodge-Tate (in particular not modular). However, the freedom of the coefficient ring allows us to prove the existence of modular points, which is the second main result of the present article (see Theorem 5.1 for the precise statement).

Theorem. *Let p be a prime, \mathcal{O} the ring of integers of a finite extension K/\mathbb{Q}_p with ramification degree $e > 1$ and π its local uniformizer. Let $\rho_n : G_{\mathbb{Q}} \rightarrow \mathrm{GL}_2(\mathcal{O}/\pi^n)$ be a continuous representation satisfying*

- ρ_n is odd.
- $\mathrm{Im}(\rho_n)$ contains $\mathrm{SL}_2(\mathcal{O}/p)$ if $n \geq e$ and $\mathrm{SL}_2(\mathcal{O}/\pi^n)$ otherwise.
- ρ_n is ordinary at p .

Let P be a set of primes containing the ramification set of ρ_n , and for each $v \in P$ pick a local deformation $\rho_v : G_v \rightarrow \mathrm{GL}_2(\mathcal{O})$ lifting $\rho_n|_{G_v}$. Then there exists a finite set of primes Q and a continuous representation $\rho : G_{P \cup Q} \rightarrow \mathrm{GL}_2(\mathcal{O})$ such that

- ρ lifts ρ_n , i.e. $\rho \equiv \rho_n \pmod{\pi^n}$.
- ρ is modular.
- For every $v \in P$, $\rho|_{I_v} \simeq \rho_v|_{I_v}$.
- For every $q \in Q$ $\rho|_{I_q}$ is unipotent and all but possibly one prime of Q satisfy $q \not\equiv \pm 1 \pmod{p}$.
- ρ is ordinary at p .

The strategy to prove both theorems is similar to the one in [Ram02] and [CP14]. We will construct, for each prime $v \in P$, a set of deformations of $\rho_n|_{G_v}$ to \mathcal{O} which contains ρ_v and a subspace N_v preserving its reductions in the sense of Proposition 2.1. Also, for $v = p$, we take C_p to be the set of nearly ordinary deformations, which will give a larger subspace N_p and therefore a smaller codomain for the morphism (1). Given this local setting, and after some manipulation of the groups appearing in (1) (and its analogue for $H^2(G_T, Ad^0 \bar{\rho})$) we are able to make such map surjective (with a 1-dimensional kernel) and the corresponding map for $H^2(G_T, Ad^0 \bar{\rho})$ injective. This implies that, with enough local conditions, the problem of lifting $\bar{\rho}$ is unobstructed, which gives Theorem 4.13.

The tricky part here is that, differently from what happens in [CP14], in some cases the subspaces N_v preserve the reductions modulo π^n of the elements of C_v , not for all n but for n bigger than a certain integer α . To overcome this situation we will, following the ideas of [KLR05], lift by adding one set of auxiliary primes for each power of π , until we reach the lift modulo π^α for the main method to work. In this way we will get Theorem 4.1.

Theorem 5.1 follows from studying the possible modular points appearing in the universal ring provided by Theorem 4.13. Notice that we will obtain a modular lift of ρ_n each time we find a nearly ordinary lift ρ such that the characters appearing in the diagonal of $\rho|_{I_p}$ can be written as an integral power of the cyclotomic character times a character of finite order.

We would like to remark that the method employed for the proof of Theorem 4.1 seems to generalize well to base fields other than \mathbb{Q} . However, the more interesting question of getting an analog to Theorem 5.1 seems to be of higher difficulty. We currently have some work in progress towards this direction. As a final remark, the results obtained in this work have overlap with the ones in [KR14] (in particular Theorem 4.1 is analog to Theorem 11 of [KR14]). Both works were independent in their first versions and in the present one we applied some of their results in order to remove hypotheses. Also, the methods of section 5 of [KR14] allow one to get the stronger Corollary 5.2.

The article is organized as follows: Section 2 concerns with the construction of the sets C_q and subspaces N_q for primes $q \neq p$. In Section 3 we do the same for the nearly ordinary condition at p . Section 4 treats global arguments for lifting and proves the first main theorem of this work. It is divided into two subsections, one for exponents at which we have C_q and N_q for every $q \in P$ and one for the ones at which we do not. In Section 5 we prove the other main theorem which concerns modularity.

1.1. Notation and conventions. In this article p will denote a rational prime, \mathcal{O} the ring of integers of a ramified finite extension K/\mathbb{Q}_p , π its local uniformizer and e its ramification degree. We will denote the residual field \mathcal{O}/π by \mathbb{F} . For a prime q we denote by G_q the local absolute Galois group $\text{Gal}(\overline{\mathbb{Q}_q}/\mathbb{Q}_q)$ and σ and τ stand for a Frobenius element and a generator of the tame inertia group of G_q . We denote

$$e_1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad e_2 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad \text{and} \quad e_3 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

which form a basis for the space of 2×2 matrices with trace 0. Also, given a prime q , d_i will stand for $\dim H^i(G_q, \text{Ad}^0 \bar{\rho})$ for $i = 0, 1, 2$. The cyclotomic character will be denoted by χ . All our deformations will have fixed determinant. By $\rho_n : G_{\mathbb{Q}} \rightarrow \text{GL}_2(\mathcal{O}/\pi^n)$ we denote a continuous Galois representation, $\bar{\rho} : G_{\mathbb{Q}} \rightarrow \text{GL}_2(\mathbb{F})$ denotes its residual representation, and by ρ we denote a representation with image in $\text{GL}_2(\mathcal{O})$. Finally v stands for the valuation in \mathcal{O} satisfying $v(\pi) = 1$.

For the definitions and main results of deformation theory, we refer to [Maz89].

1.2. Acknowledgments. We would like to thank Ariel Pacetti for many useful conversations, and to Ravi Ramakrishna for many remarks and suggestions in a draft version of the present article.

2. LOCAL DEFORMATION THEORY AT $q \neq p$

Let $q \neq p$ a prime and let $\rho : G_q \rightarrow \text{GL}_2(\mathcal{O})$ be a continuous representation. We denote by $\bar{\rho}$ its reduction mod π . We know that the elements of $H^1(G_q, \text{Ad}^0 \bar{\rho})$ act on the deformations $\tilde{\rho}$ with coefficients in \mathcal{O}/π^m by $u \cdot \tilde{\rho} = (1 + \pi^{m-1}u)\tilde{\rho}$. Recall the notion of an element of $H^1(G_q, \text{Ad}^0 \bar{\rho})$ preserving a set of deformations.

Definition. Let C_q be a set of deformations $\{\rho : G_q \rightarrow \text{GL}_2(\mathcal{O})\}$ and $u \in H^1(G_q, \text{Ad}^0 \bar{\rho})$. We say that u preserves C_q if for any $\tilde{\rho} : G_q \rightarrow \text{GL}_2(\mathcal{O}/\pi^m)$ which is the reduction of a deformation of C_q we have that $u \cdot \tilde{\rho}$ is also the reduction of some deformation of C_q .

We want to prove that for every ρ there exists a set C_q of deformations containing it that is preserved by a subspace N_q of certain dimension. We can prove such set exists for almost all possible ρ but some particular ones.

Definition. We say that a representation $\rho : G_q \rightarrow \text{GL}_2(\mathcal{O})$ is “bad” if

$$\rho \simeq \begin{pmatrix} \psi_1 & \frac{\psi_1 - \psi_2}{\pi^r} \\ 0 & \psi_2 \end{pmatrix}$$

over $\text{GL}_2(\mathcal{O})$ with $r > 0$ and moreover the following holds:

- $\bar{\rho}$ is unramified and $\bar{\rho}^{ss}(\sigma)$ is scalar.
- $v(\psi_1(\sigma) - \psi_2(\sigma)) < v(\frac{\psi_1(\tau) - \psi_2(\tau)}{\pi^r})$.

We currently do not know how to find the desired family of deformations in these cases. Observe that finding the families C_q and subspaces N_q accounts for proving that the Balancedness Assumption of [KR14] holds (see Assumption before Definition 3). In that scenario, if we start with a mod π^n deformation ρ_n and a prime q for which every lift to \mathcal{O} is bad, we do not know how to prove that the Assumption holds.

Proposition 2.1. *Let $\rho : G_q \rightarrow \mathrm{GL}_2(\mathcal{O})$ be a continuous representation. If ρ is not bad then there always exists a positive integer α , a set C_q of deformations of the reduction of ρ modulo π^α to characteristic 0 that contains ρ (up to $\mathrm{GL}_2(\mathcal{O})$ -isomorphism), and a subspace $N_q \subseteq H^1(G_q, \mathrm{Ad}^0 \bar{\rho})$ such that:*

- All the elements of C_q are isomorphic over $\mathrm{GL}_2(\mathcal{O})$ when restricted to inertia.
- $N_q \subseteq H^1(G_q, \mathrm{Ad}^0 \bar{\rho})$ has codimension equal to $\dim H^2(G_q, \mathrm{Ad}^0 \bar{\rho})$.
- Every $u \in N_q$ preserves the mod π^m reductions of elements of C_q for $m \geq \alpha$.

In other words, there exists a smooth deformation condition containing ρ of dimension equal to $\dim H^1(G_q, \mathrm{Ad}^0 \bar{\rho}) - \dim H^2(G_q, \mathrm{Ad}^0 \bar{\rho})$ if we start lifting from a big enough exponent.

In order to prove the proposition, let us recall the main results about types of deformations and types of reduction mod π . Although they are mainly known, the results and its proofs are contained in section 2 of [CP14].

Proposition 2.2. *Let $q \neq 2$, be a prime number, with $q \neq p$. Then every representation $\rho : G_q \rightarrow \mathrm{GL}_2(\mathbb{F})$, up to twist by a character of finite order, belongs to one of the following three types:*

- **Principal Series:** $\rho \simeq \begin{pmatrix} \phi & 0 \\ 0 & 1 \end{pmatrix}$ or $\rho \simeq \begin{pmatrix} 1 & \psi \\ 0 & 1 \end{pmatrix}$.
- **Steinberg:** $\rho \simeq \begin{pmatrix} \chi & \mu \\ 0 & 1 \end{pmatrix}$, where $\mu \in H^1(G_q, \mathbb{F}(\chi))$ and $\mu|_{I_q} \neq 0$.
- **Induced:** $\rho \simeq \mathrm{Ind}_{G_M}^{G_q}(\xi)$, where M/\mathbb{Q}_q is a quadratic extension and $\xi : G_M \rightarrow \mathbb{F}^\times$ is a character not equal to its conjugate under the action of $\mathrm{Gal}(M/\mathbb{Q}_q)$.

Here $\phi : G_q \rightarrow \mathbb{F}^\times$ is a multiplicative character and $\psi : G_q \rightarrow \mathbb{F}$ is an unramified additive character.

Proposition 2.3. *Let $\tilde{\rho} : G_q \rightarrow \mathrm{GL}_2(\overline{\mathbb{Z}_p})$ be a continuous representation. Then up to twist (by a finite order character times and unramified character) and $\mathrm{GL}_2(\overline{\mathbb{Z}_p})$ equivalence we have:*

- **Principal Series:** $\tilde{\rho} \simeq \begin{pmatrix} \phi & \pi^n(\phi-1) \\ 0 & 1 \end{pmatrix}$, with $n \in \mathbb{Z}_{\leq 0}$ satisfying $\pi^n(\phi-1) \in \overline{\mathbb{Z}_p}^\times$ or $\tilde{\rho} \simeq \begin{pmatrix} 1 & \psi \\ 0 & 1 \end{pmatrix}$.
- **Steinberg:** $\tilde{\rho} \simeq \begin{pmatrix} \chi & \pi^n \mu \\ 0 & 1 \end{pmatrix}$, with $n \in \mathbb{Z}_{\geq 0}$.
- **Induced:** There exists a quadratic extension M/\mathbb{Q}_q and a character $\xi : G_M \rightarrow \overline{\mathbb{Z}_p}^\times$ not equal to its conjugate under the action of $\mathrm{Gal}(M/\mathbb{Q}_q)$ such that $\tilde{\rho} \simeq \langle v_1, v_2 \rangle_{\mathcal{O}_L}$, where for α a generator of $\mathrm{Gal}(M/\mathbb{Q}_p)$ and $\beta \in G_M$, the action is given by

$$\beta(v_1) = \xi(\beta)v_1, \quad \beta(v_2) = \xi^\alpha(\beta)v_2, \quad \alpha(v_1) = v_2 \quad \text{and} \quad \alpha(v_2) = \xi(\alpha^2)v_1,$$

or

$$\tilde{\rho}(\beta) = \begin{pmatrix} \xi(\beta) & \frac{\xi(\beta) - \xi^\alpha(\beta)}{\xi^{\pi^n}(\beta)} \\ 0 & \xi^{\pi^n}(\beta) \end{pmatrix} \quad \text{and} \quad \tilde{\rho}(\alpha) = \begin{pmatrix} -a & \frac{\xi(\alpha^2) - a^2}{\pi^n} \\ \pi^n & a \end{pmatrix}$$

where ξ^α is the character of G_M defined by $\xi^\alpha(g) = \xi(\alpha g \alpha^{-1})$ and $a \in \mathcal{O}_L^\times$. Observe that when M/\mathbb{Q} is ramified we can take α and β to be a Frobenius element and a generator of the tame inertia respectively.

Proposition 2.4. *Let $\tilde{\rho}$ be as above and $\bar{\rho}$ its mod p reduction. We have the following types of reduction:*

- If $\tilde{\rho}$ is Principal Series, then $\bar{\rho}$ can be Principal Series or Steinberg, and the latter occurs only when $q \equiv 1 \pmod{p}$.
- If $\tilde{\rho}$ is Steinberg, then $\bar{\rho}$ can be Steinberg or Principal Series, and the latter occurs only when $\tilde{\rho}$ is unramified.
- If $\tilde{\rho}$ is Induced, then $\bar{\rho}$ can be Induced, Steinberg or unramified Principal Series. For the last two cases we must have $q \equiv -1 \pmod{p}$ and M/\mathbb{Q} ramified.

To prove Proposition 2.1 we consider all the possible pairs of $\mathrm{GL}_2(\mathbb{F})$ and $\mathrm{GL}_2(\mathcal{O})$ -equivalence classes for ρ and $\bar{\rho}$ (indexing them first by the class of $\bar{\rho}$), and for each of them we define the corresponding deformation class and cohomology subspace.

• **Case 1: $\bar{\rho}$ is ramified Principal Series.** Proposition 2.4 implies that a mod π Principal Series can only come from a characteristic 0 principal series. The full study of this case is done in Case 1 of Section 4 of [CP14]. The work there is done for \mathcal{O} unramified but the same applies in our situation.

• **Case 2: $\bar{\rho}$ is Steinberg.** When $\bar{\rho}$ is Steinberg, it can be the reduction of any of the three characteristic 0 ramified types:

• **Case 2.1: ρ is Steinberg.** The definition of C_q and N_q is essentially the same as [CP14], Case 2 of Section 4. Although in that work only the case where \mathcal{O}/\mathbb{Z}_p is unramified is treated, the ramification of \mathcal{O} does not affect the results.

• **Case 2.2: ρ is Principal Series.** Proposition 2.4 implies that $q \equiv 1 \pmod{p}$. Let $\rho = \begin{pmatrix} \psi & * \\ 0 & 1 \end{pmatrix}$. Without loss of generality, we can take $*(\tau) = 1$. We have the following lemma:

Lemma 2.5. *A deformation $\tilde{\rho} : G_q \rightarrow \mathrm{GL}_2(\mathcal{O}/\pi^m)$ which has the form:*

$$\tilde{\rho}(\tau) = \begin{pmatrix} \psi(\tau) & 1 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad \tilde{\rho}(\sigma) = \begin{pmatrix} \alpha & \gamma \\ 0 & \beta \end{pmatrix},$$

has a unique lift to characteristic zero of the same form if and only if $\beta^2 + \gamma(\psi(\tau) - 1)\beta + \Psi \equiv 0 \pmod{\pi^m}$, where Ψ is a fixed lift determinant.

Proof. This Lemma is part of a computation made in Proposition 3.4 of [Kha06]. There, it is done for deformations with coefficients in unramified coefficient field and its mod p reductions, but the same proof works in general. \square

Let $j \in H^1(G_q, \mathrm{Ad}^0 \bar{\rho})$ be the element defined by:

$$j(\sigma) = e_2, \quad j(\tau) = 0,$$

and take N_q to be the subspace it generates. Also let C_q be the set of deformations to \mathcal{O} which have the form given in Lemma 2.5. Observe that any mod π^m reduction of an element of C_q satisfies the equation $\beta^2 + \gamma(\psi(\tau) - 1)\beta + \Psi = 0$, and acting by j on it does not affect this (as π divides $\psi(\tau) - 1$ so adding a multiple of π^{m-1} to γ does not change the equation modulo π^m). Then, Lemma 2.5 guarantees that C_q and N_q satisfy the property we are looking for.

• **Case 2.3: ρ is Induced.** Proposition 2.4 tells us that necessarily $q \equiv -1 \pmod{\pi}$ and by results in Section 3 of [CP14] we have that $d_1 = d_2 = 1$. Therefore we can take $N_q = \{0\}$ and $C_q = \{\rho\}$.

• **Case 3: $\bar{\rho}$ is Induced.** By Proposition 2.4, when $\bar{\rho}$ is Induced, ρ must be Induced as well. The choice of C_q and N_q in this case is explained in Case 3 of Section 4 of [CP14].

• **Case 4: $\bar{\rho}$ is unramified.** Being unramified, $\bar{\rho}$ allows lifts to any type of deformation. We must treat each of them separately as they have many subcases. The case of ρ being **Steinberg** is dealt with in Case 4 of Section 4 of [CP14]. There are two other cases left to study. In each of them we will distinguish between three types of equivalence classes for $\bar{\rho}$, according to the image of Frobenius. This case is the one that includes bad primes, the calculations made here show where the badness condition appears.

Case 4.1: ρ is Principal Series. In this case we have $\rho = \begin{pmatrix} \phi & \pi^{-r}(\phi-1) \\ 0 & 1 \end{pmatrix}$ with $r \geq 0$. By Proposition 2.4 we necessarily have $q \equiv 1 \pmod{\pi}$.

· If $\bar{\rho}(\sigma) = \begin{pmatrix} \beta & 0 \\ 0 & 1 \end{pmatrix}$ with $\beta \neq 1$ we have $d_1 = 2$ and $d_2 = 1$ so we are looking for a one-dimensional subspace N_q . Observe that necessarily $\rho \simeq \begin{pmatrix} \phi & 0 \\ 0 & 1 \end{pmatrix}$ over \mathcal{O} as $\psi(\sigma) \equiv \beta \not\equiv 1 \pmod{\pi}$ so $(\psi - 1)/\pi$ is not an integer.

Let $u \in H^1(G_q, Ad^0 \bar{\rho})$ be the cocycle defined by $u(\sigma) = e_1$ and $u(\tau) = 0$. We can take $N_q = \langle u \rangle$ and C_q the set of representations of the form $\rho \simeq \begin{pmatrix} \phi\psi & 0 \\ 0 & \psi^{-1} \end{pmatrix}$ for $\psi : G_q \rightarrow \mathcal{O}^\times$ unramified. It is easily checked that these satisfy the desired properties.

· If $\bar{\rho}(\sigma_q) = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ the corresponding dimensions are $d_1 = 2$ and $d_2 = 1$. In this case we have $\rho \simeq \begin{pmatrix} \phi & \frac{\phi-1}{\pi^r} \\ 0 & 1 \end{pmatrix}$, with $v_\pi(\phi - 1) = r$. Now take the cocycle defined by $u(\sigma) = 0$ and $u(\tau) = e_2$. We take $\alpha = v(x - y) + 2$ and set $N_q = \langle u \rangle$ and C_q the set of deformations of $\bar{\rho}$ such that $\rho \simeq \begin{pmatrix} \gamma\phi & \beta \frac{\gamma\phi - \gamma^{-1}}{\pi^\alpha} \\ 0 & \gamma^{-1} \end{pmatrix}$ for some $\beta \in \mathcal{O}^\times$ and γ an unramified character congruent to 1 modulo π^α .

Lemma 2.6. *The set C_q and subspace N_q defined above satisfy that N_q preserves C_q mod π^m for all m such that ϕ is ramified mod π^m .*

Proof. Assume that $\rho(\sigma) = \begin{pmatrix} a & \frac{a-b}{\pi^r} \\ 0 & b \end{pmatrix}$ and $\rho(\tau) = \begin{pmatrix} x & \frac{x-y}{\pi^r} \\ 0 & y \end{pmatrix}$. Our assumptions on ρ not being bad tell us that $v(a - b) \geq v((x - y)/\pi^r)$. If we have $v(a - b) > v(x - y)$ then we change σ by $\tau\sigma$ (which is another Frobenius element). In this way, we can assume that $v(a - b) \leq v(x - y)$.

We want to prove that if we have a deformation $\tilde{\rho} : G_q \rightarrow GL_2(\mathcal{O}/\pi^m)$ that sends

$$\tilde{\rho}(\sigma) = \begin{pmatrix} a & c \\ 0 & b \end{pmatrix} \quad \text{and} \quad \tilde{\rho}(\tau) = \begin{pmatrix} x & z \\ 0 & y \end{pmatrix},$$

with $x \neq y$, and is the reduction of some element of C_q (i.e. $c = \beta(a - b)$ and $z = \beta(x - y)$ for some $\beta, a, b, c, x, y, z \in \mathcal{O}$), then $u \cdot \tilde{\rho}$ is also the reduction of some element of C_q . Recall that

$$u \cdot \tilde{\rho}(\sigma) = \begin{pmatrix} a & c \\ 0 & b \end{pmatrix} \quad \text{and} \quad u \cdot \tilde{\rho}(\tau) = \begin{pmatrix} x & z + \pi^{m-1} \\ 0 & y \end{pmatrix}.$$

It is easily checked that the cocycle v that sends σ to $\lambda_1 e_1 + \lambda_2 e_2$ and τ to 0 is a coboundary for any choice of $\lambda_1, \lambda_2 \in \mathbb{F}$. So $\tilde{\rho}$ can also be thought as

$$u \cdot \tilde{\rho}(\sigma) = \begin{pmatrix} a(1 + \lambda_1 \pi^{m-1}) & c + \lambda_2 \pi^{m-1} \\ 0 & b(1 + \lambda_1 \pi^{m-1})^{-1} \end{pmatrix} \quad \text{and} \quad u \cdot \tilde{\rho}(\tau) = \begin{pmatrix} x & z + \pi^{m-1} \\ 0 & y \end{pmatrix}$$

for any $\lambda_1, \lambda_2 \in \mathbb{F}$. Therefore, it is enough to find some $\lambda_1, \lambda_2 \in \mathcal{O}$ such that

$$\frac{x - y}{z + \pi^{m-1}} = \frac{a(1 + \lambda_1 \pi^{m-1}) - b(1 + \lambda_1 \pi^{m-1})^{-1}}{c + \lambda_2 \pi^{m-1}}.$$

given that

$$\frac{x - y}{z} = \frac{a - b}{c}.$$

Expanding this equation we find out that it is equivalent to

$$\lambda_1(z + \pi^{m-1})(a + b(1 + \lambda_1 \pi^{m-1})^{-1}) + a - b = \lambda_2(x - y).$$

which has a solution given that $v(a - b) \geq v(z)$ (we solve first for λ_1 in order for both sides to have the same valuation, and then there is a λ_2 that makes the equality true). \square

Remark. Observe that when the condition $v(a - b) \geq v(z)$ does not hold, the last equation of the proof does not have a solution, since no matter which $\lambda_1, \lambda_2 \in \mathcal{O}$ we pick, the valuation of the left hand side is $v(a - b)$ and the valuation of the right hand side is bigger or equal than $v(x - y) \geq v(z) > v(a - b)$. Moreover, following the same type of computations we can prove that for the chosen set C_q there is no non-trivial $u \in H^1(G_q, Ad^0 \bar{\rho})$ such that the subspace $\langle u \rangle$ preserves C_q .

· If $\bar{\rho}(\sigma_q) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ we have $d_1 = 6$ and $d_2 = 3$ therefore we need to find a subspace N_q of dimension 3. This case is a little more involved than the other two as there are non-trivial elements of $H^1(G_q, Ad^0 \bar{\rho})$ that act trivially on modulo π^m deformations for high powers of π . We follow the same ideas as in the study of the Steinberg-reducing-to-unramified case (the spirit of these ideas is taken from the approach to trivial primes followed in [HR08]).

Assume first that $\rho \simeq \begin{pmatrix} \phi & 0 \\ 0 & 1 \end{pmatrix}$. We will take C_q as the set of representations of the form

$$\rho' \simeq \begin{pmatrix} \gamma\phi & 0 \\ 0 & \gamma^{-1} \end{pmatrix}$$

with $\gamma : G_q \rightarrow \mathcal{O}^\times$ an unramified character, that lift the reduction mod π^α of ρ , with $\alpha = v(\phi_1(\tau) - \phi_2(\tau)) + 2$. Clearly, the set C_q is preserved by the cocycle u_1 that sends $\sigma \mapsto e_1$ and $\tau \mapsto 0$.

We will construct two more cocycles u_2 and u_3 that act trivially on reductions modulo π^m of deformations on C_q for $m \geq \alpha$. Let $\tilde{\rho} : G_q \rightarrow GL_2(\mathcal{O}/\pi^m)$ be the mod π^m reduction of an element in C_q . Let $\tilde{\rho}(\sigma) = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}$ and $\tilde{\rho}(\tau) = \begin{pmatrix} x & 0 \\ 0 & y \end{pmatrix}$. Let $u \in H^1(G_q, Ad^0 \bar{\rho})$. To prove that u acts trivially on $\tilde{\rho}$ we need to find a matrix $C \in GL_2(\mathcal{O}/\pi^m)$ such that $C \equiv Id \pmod{\pi}$ and $C\tilde{\rho}C^{-1} = (Id + \pi^{m-1}u)\tilde{\rho}$. One can find such matrix by taking $C = \begin{pmatrix} 1+\pi\alpha & \pi\beta \\ \pi\gamma & 1+\pi\delta \end{pmatrix}$ and explicitly computing $C\tilde{\rho} = (Id + \pi^{m-1}u)\tilde{\rho}C$ at σ and τ . In this way, one finds out that if $v(x-y) > v(a-b)$ then the cocycles u_2 and u_3 sending σ to e_2 and e_3 respectively and τ to 0 act trivially on the reductions of elements of C_q . The corresponding base change matrices C_i that conjugate $(Id + \pi^{m-1}u_i)\tilde{\rho}$ into $\tilde{\rho}$ are

$$C_2 = \begin{pmatrix} 1 & \frac{\pi^{m-1}}{a-b} \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad C_3 = \begin{pmatrix} 1 & 0 \\ -\frac{\pi^{m-1}}{a-b} & 1 \end{pmatrix}.$$

It remains to check what happens when $v(\phi_1(\tau) - \phi_2(\tau)) \leq v(\phi_1(\sigma) - \phi_2(\sigma))$. Observe that we can always assume that $v(\phi_1(\tau) - \phi_2(\tau)) = v(\phi_1(\sigma) - \phi_2(\sigma))$ in this case, by simply changing σ for $\tau\sigma$. In this case we can take u_2 sending σ to e_2 and τ to λe_2 and u_3 sending σ to e_3 and τ to λe_3 for $\lambda = (x-y)/(a-b) \in \mathbb{F}$ (notice that this does not depend on $\tilde{\rho}$). Again, the action of both cocycles will be trivial and the base change matrices will be the same as before.

It remains to consider the case where $\rho \simeq \begin{pmatrix} \phi & (\phi-1)/\pi^r \\ 0 & 1 \end{pmatrix}$ for $r > 0$. Let $\alpha = v((\phi-1)/\pi^r) + 2$ and let C_q be the set of deformations of the mod π^α reduction of ρ such that $\rho \simeq \begin{pmatrix} \gamma\phi & \beta \frac{\gamma\phi-\gamma^{-1}}{\gamma^{-1}} \\ 0 & \gamma^{-1} \end{pmatrix}$ for some $\beta \in \mathcal{O}^\times$ and γ an unramified character congruent to 1 modulo π^α . By doing the exact same calculation as in Lemma 2.6 it can be proved that the cocycle u_1 that sends σ to 0 and τ to e_2 preserves the reductions of elements in C_q , given that ρ is not bad. We still need two more elements preserving C_q . As in the previous case, we have two cocycles that act trivially on mod π^m reductions of elements of C_q for $m \geq \alpha$. Let $\tilde{\rho}$ is a mod π^m reduction of some element in C_q given by $\tilde{\rho}(\sigma) = \begin{pmatrix} a & c \\ 0 & b \end{pmatrix}$ and $\tilde{\rho}(\tau) = \begin{pmatrix} x & z \\ 0 & y \end{pmatrix}$.

If $v(a-b) < v(x-y)$, we take u_2 sending σ to e_1 and τ to 0 and u_3 the one that sends σ to e_2 and τ to 0. We claim that these act trivially on $\tilde{\rho}$ if $m \geq v(z) + 2$.

If otherwise $v(a-b) \geq v(x-y)$ we can assume that $v(a-b) = v(x-y)$ as in Lemma 2.6. Let

$$\lambda = \frac{x-y}{a-b} = \frac{z}{c}.$$

Let u_2 be the cocycle that sends σ to e_1 and τ to λe_1 and u_3 the one that sends σ to e_2 and τ to λe_2 . These act trivially on $\tilde{\rho}$ if $m \geq v(z) + 2$.

It can be checked that in both cases the base change matrices given by

$$C_2 = \begin{pmatrix} 1 & 0 \\ -\frac{\pi^{m-1}}{c} & 1 \end{pmatrix} \quad \text{and} \quad C_3 = \begin{pmatrix} 1 + \frac{\pi^{m-1}}{c} & 0 \\ 0 & 1 \end{pmatrix}$$

serve to prove the trivialness of the action of u_2 and u_3 respectively. This concludes the case.

Case 4.2: ρ is Induced. Proposition 2.4 says that whenever ρ is induced and $\bar{\rho}$ is unramified, $\bar{\rho}(\sigma)$ has eigenvalues 1 and -1 (up to twist) and $q \equiv -1 \pmod{p}$. In this case we have that

$d_1 = 3$ and $d_2 = 2$. We want to find a set C_q and a subspace N_q of dimension 1 preserving it. Let $C_q = \{\rho\}$. As in the Principal Series case, we will be able to find non trivial cocycles that act trivially on mod π^m reductions of ρ for m big enough. We split into the two possible families of $\mathrm{GL}_2(\overline{\mathbb{Z}_p})$ -equivalence classes of induced representations given by Proposition 2.3.

· If $\rho(\sigma) = \begin{pmatrix} 0 & t \\ 1 & 0 \end{pmatrix}$ and $\rho(\tau) = \begin{pmatrix} x & 0 \\ 0 & y \end{pmatrix}$, it can be checked that the cocycle u sending σ to 0 and τ to $e_2 - e_3$ is non trivial. The cocycle u acts trivially modulo π^m for all $m \geq v(x-y) + 2$ whenever $v(t-1) > v(x-y)$. In this case the base change matrix is given by

$$C = \begin{pmatrix} 1 & -\frac{\pi^{m-1}}{x-y} \\ -\frac{\pi^{m-1}}{x-y} & 1 \end{pmatrix}.$$

If we are in a case where $v(t-1) \leq v(x-y)$ then we can go back to a case where $v(t-1) > v(x-y)$ by twisting ρ . Let $\eta \in 1 + \pi\mathcal{O}$ be an unit such that $v(\eta^2 t - 1) > v(x-y)$ and $\gamma : G_q \rightarrow \mathcal{O}^\times$ an unramified character mapping σ to η . If we twist ρ by γ then the deformation obtained is equivalent to ρ' sending

$$\rho'(\sigma) = \begin{pmatrix} 0 & \eta^2 t \\ 1 & 0 \end{pmatrix} \quad \text{and} \quad \rho'(\tau) = \begin{pmatrix} x & 0 \\ 0 & y \end{pmatrix}$$

via the base change matrix

$$C = \begin{pmatrix} \eta & 0 \\ 0 & 1 \end{pmatrix}.$$

· If $\rho(\sigma) = \begin{pmatrix} -a & c \\ b & a \end{pmatrix}$ and $\rho(\tau) = \begin{pmatrix} x & z \\ 0 & y \end{pmatrix}$, as in the previous case, it can be checked that the cocycle u that sends σ to 0 and τ to e_2 is non trivial. Again, the action of this cocycle in the reduction modulo π^m of ρ is trivial for $m \geq v(z) + 2$. The base change matrix that works for this case is

$$C = \begin{pmatrix} 1 + \frac{\pi^{m-1}}{\frac{c\pi^{m-1}}{2az}} & \frac{b\pi^{m-1}}{2az} \\ \frac{c\pi^{m-1}}{2az} & 1 \end{pmatrix}.$$

3. LOCAL DEFORMATION THEORY AT p

At the prime p we will impose the deformation condition of being “nearly ordinary” (as in [CM09]). This section is mainly about gathering previously done calculations, and all the deformations appearing are deformations of the local Galois group G_p .

Definition. We say that a deformation of G_p is “nearly ordinary” if its restriction to the inertia subgroup is upper-triangular and its semisimplification is not scalar, i.e. if

$$\rho|_{I_p} = \begin{pmatrix} \psi_1 & * \\ 0 & \psi_2 \end{pmatrix}.$$

with $\psi_1 \neq \psi_2$.

We will prove the following theorem.

Theorem 3.1. *Let $\rho_n : G_p \rightarrow \mathcal{O}/\pi^n$ be a nearly ordinary deformation and $\overline{\rho}$ its mod π reduction. There is a family of nearly ordinary deformations C_p to characteristic 0 such that ρ_n is the reduction of a member of C_p and a subspace $N_p \subseteq H^1(G_p, \mathrm{Ad}^0 \overline{\rho})$ of codimension equal to $\dim H^2(G_p, \mathrm{Ad}^0 \overline{\rho}) + 1$ preserving C_p in the sense of Proposition 2.1.*

Proof. Let

$$\overline{\rho} \simeq \begin{pmatrix} \psi_1 & * \\ 0 & \psi_2 \end{pmatrix}.$$

By twisting $\overline{\rho}$ by ψ_2^{-1} we can assume that $\psi_2 = 1$. Let U be the set of upper-triangular 2×2 matrices of trace 0. To prove the theorem we will construct for each possible $\overline{\rho}$, the corresponding set and subspace and verify that its dimensions satisfy the statement of the theorem. In most of the cases C_p will consist on all nearly ordinary deformations of $\overline{\rho}$ and N_p will be the image of $H^1(G_p, U)$ in $H^1(G_p, \mathrm{Ad}^0 \overline{\rho})$.

In order to see this, we simply compute the dimension of the image of $H^1(G_p, U)$ in $H^1(G_p, Ad^0 \bar{\rho})$ and compare it with $\dim H^1(G_p, Ad^0 \bar{\rho}) - \dim H^2(G_p, Ad^0 \bar{\rho}) - 1$. We can reduce the computation of all these values to finding the dimensions of $H^0(G_p, U)$, $H^0(G_p, U^*)$, $H^0(G_p, Ad^0 \bar{\rho})$ and $H^0(G_p, (Ad^0 \bar{\rho})^*)$ by using local Tate duality and the formula for the Euler-Poincare characteristic (which is equal to 2 for U and 3 for $Ad^0 \bar{\rho}$). The kernel of the map $H^1(G_p, U) \rightarrow H^1(G_p, Ad^0 \bar{\rho})$ induced by $U \subseteq Ad^0 \bar{\rho}$ is contained in $H^0(G_p, Ad^0 \bar{\rho}/U)$, which is equal to 0 given that $\psi_1 \neq \psi_2$. With all these tools, the required dimensions are easily computed and we obtain that taking C_p as the set of all nearly ordinary deformations and N_p as the image of $H^1(G_p, U)$ in $H^1(G_p, Ad^0 \bar{\rho})$ works for all cases but the one in which $\bar{\rho}$ is decomposable and ψ_1 is the cyclotomic character (recall we are assuming $\psi_2 = 1$).

In this case the universal ring for nearly ordinary deformations is not smooth and C_p is not preserved by N_p as there are some mod π^s nearly ordinary deformations of $\bar{\rho}$ that do not lift back to characteristic 0. We need to take a smaller set C_p in this case. In order to solve this, we claim that the universal deformation ring for nearly ordinary lifts of $\bar{\rho}$ with fixed determinant ψ is isomorphic to the universal deformation ring for **ordinary** lifts of $\bar{\rho}$ with arbitrary determinant. To see this, just observe that from any nearly ordinary deformation of $\bar{\rho}$ to a ring A we can obtain an ordinary lift of $\bar{\rho}$ by twisting by inverse of the character appearing in the place (2, 2). To go the other way round, if we have an ordinary deformation $\tilde{\rho}$ of $\bar{\rho}$ to A given by

$$\tilde{\rho} = \begin{pmatrix} \omega_1 & * \\ 0 & \omega_2 \end{pmatrix}$$

where ω_2 is unramified and want to obtain a nearly ordinary deformation of $\bar{\rho}$ with determinant ψ , we need to twist by a square root of $\psi(\omega_1 \omega_2)^{-1}$. This character has a square root by Hensel's lemma, as its reduction modulo the maximal ideal is 1 which has a square root. Given this identification, the wanted result follows from Proposition 6 of [KR14], where the same result is proven for the ordinary arbitrary determinant case. \square

4. GLOBAL DEFORMATION THEORY

In this section we prove the one of the main results of this article.

Theorem 4.1. *Let $n \geq 2$ be an integer and $\rho_n : G_{\mathbb{Q}} \rightarrow \mathrm{GL}_2(\mathcal{O}/\pi^n)$ be a continuous representation which is odd and nearly ordinary at p . Assume that $\mathrm{Im}(\rho_n)$ contains $\mathrm{SL}_2(\mathcal{O}/p)$ if $n \geq e$ and $\mathrm{SL}_2(\mathcal{O}/\pi^n)$ otherwise. Let P be a set of primes of \mathbb{Q} containing the ramification set of ρ_n . For each $v \in P \setminus \{p\}$ fix a local deformation $\rho_v : G_v \rightarrow \mathrm{GL}_2(\mathcal{O})$ that lifts $\rho_n|_{G_v}$ and is not bad (see Definition 2). Then there exists a continuous representation $\rho : G_{\mathbb{Q}} \rightarrow \mathrm{GL}_2(\mathcal{O})$ and a finite set of primes R such that:*

- ρ lifts ρ_n , i.e. $\rho \equiv \rho_n \pmod{\pi^n}$.
- ρ is unramified outside $P \cup R$.
- For every $v \in P$, $\rho|_{I_v} \simeq \rho_v|_{I_v}$ over $\mathrm{GL}_2(\mathcal{O})$.
- ρ is nearly ordinary at p .
- All the primes of R , except possibly one, are not congruent to 1 modulo p .

The proof of this theorem essentially consists on finding a way to lift ρ_n to characteristic 0 one power of π at a time. We will split the proof into two sections, essentially because the local results we have so far are split in two different cases depending on whether the exponent m in each step is big enough or not. Let α be the integer obtained in the following way: for each $v \in P$, Proposition 2.1 gives an integer α_v such that there is a set C_v containing ρ_v and a subspace N_v preserving its reductions mod π^n for $n \geq \alpha_v$. Let α be the maximum of the α_v 's for $v \in P$. When lifting from π^m to π^{m+1} for $m \geq \alpha$ we are in a global setting similar to the one in [Ram02]. The existence of the sets C_v and subspaces N_v let us mimic the argument given there. When working modulo m for $m < \alpha$, we do not count with these sets and subspaces for all the primes of P and therefore are unable to overcome the local obstructions in the same fashion as before. In this case, we will follow the ideas from [KLR05] and will lift to \mathcal{O}/π^α by adding a finite number of auxilliary primes at each power of π .

As \mathcal{O} is the ring of integers of a ramified extension we have that \mathcal{O}/π^2 is isomorphic to the dual numbers and therefore the projection mod π^2 of ρ_n defines an element in $H^1(G_{\mathbb{Q}}, Ad^0 \bar{\rho})$ which we will call f . Observe that our hypotheses imply that the image of the projection of ρ_n to \mathcal{O}/π^2 contains $SL_2(\mathcal{O}/\pi^2)$ and thus $f \neq 0$ as an element in $H^1(G_{\mathbb{Q}}, Ad^0 \bar{\rho})$.

4.1. Getting to mod π^α . Assume that the exponent n we start with is strictly smaller than the natural number α from Proposition 2.1 (if this is not the case we are done). The idea is to adjust the main argument of [KLR05] to our situation. Recall the following definition.

Definition. A prime q is *nice* for $\bar{\rho}$ if it satisfies the following properties

- The prime q is not congruent to $\pm 1 \pmod{p}$.
- The representation ρ_n is unramified at q .
- The eigenvalues of $\bar{\rho}(\sigma)$ have ratio q .

We say that q is nice for ρ_n if furthermore

- The eigenvalues of $\rho_n(\sigma)$ have ratio q .

At a nice prime q we consider the set

$$C_q = \{\text{deformations } \rho \text{ of } \bar{\rho}|_{G_q} : \rho(\sigma) = \begin{pmatrix} q & 0 \\ 0 & 1 \end{pmatrix}\},$$

and the subspace $N_q \subseteq H^1(G_q, Ad^0 \bar{\rho})$ generated by the cocycle u sending σ to 0 and τ to e_2 . It is easy to check that u preserves the reductions of elements of C_q .

The work on [KLR05] is based on the existence of nice primes that are either zero or non zero at certain elements of both $H^1(G_U, Ad^0 \bar{\rho})$ and $H^1(G_U, (Ad^0 \bar{\rho})^*)$ for different sets of primes U . We claim that the same arguments work in this settings except for the cases when the element $f \in H^1(G_P, Ad^0 \bar{\rho})$ attached to $\rho_n \pmod{\pi^2}$ is involved. We will sort this obstacle by adding the following primes.

Definition. We say that a prime q is a *special prime* for f if

- The representation ρ_n is unramified at q .
- $\rho_n(\sigma) = \begin{pmatrix} 1 & \pi \\ 0 & 1 \end{pmatrix}$.
- The prime $q \equiv 1 \pmod{\pi^n}$.

Note that for such primes q we have that $f|_{G_q} \neq 0$ as an element in $H^1(G_q, Ad^0 \bar{\rho})$. We need some partial results to state and prove the main result of this case (Theorem 4.7).

Lemma 4.2. *Let $f, f_1, \dots, f_r \in H^1(G_{\mathbb{Q}}, Ad^0 \bar{\rho})$ and $\phi_1, \dots, \phi_s \in H^1(G_{\mathbb{Q}}, (Ad^0 \bar{\rho})^*)$ be linearly independent sets.*

a) *Let $I \subseteq \{1, \dots, r\}$ and $J \subseteq \{1, \dots, s\}$. There is a Chebotarev set of primes v such that*

- *v is nice for ρ_n .*
- *$f_i|_{G_v} \neq 0$ if $i \in I$ and $f_i|_{G_v} = 0$ if $i \notin I$.*
- *$\phi_j|_{G_v} \neq 0$ if $j \in J$ and $\phi_j|_{G_v} = 0$ if $j \notin J$.*

b) *Also, there is a Chebotarev set of primes w such that*

- *w is a special prime for f , henceforth $f|_{G_w} \neq 0$.*
- *$f_i|_{G_w} = 0$ for all $1 \leq i \leq r$.*

Remark. For special primes we can also define a set C_v of deformations to characteristic 0 and a subspace $N_v \subseteq H^1(G_v, Ad^0 \bar{\rho})$ of codimension $\dim H^2(G_v, Ad^0 \bar{\rho})$ preserving it. This is explicitly done in Lemma 4.1 of [CP14]. The procedure is the same as the one employed in cases 4.1 and 4.2 of Section 2. We will not reproduce the results here in an effort to preserve elegance. Notice that this corresponds to the case of a trivial local $\bar{\rho}$ lifting to a Steinberg ρ .

Proof. This is a slight modification of Fact 5 of [KLR05]. The main problem with nice primes in ramified extensions is that if v is a nice prime then $f|_{G_v} = 0$. The use of special primes for f solves this problem, since almost by definition if v is a special prime, $f|_{G_v} \neq 0$. In order to check that Chebotarev conditions at the different f_i 's and ϕ_j 's are disjoint from the condition of being nice for ρ_n , and that this last condition only overlaps with extension corresponding to the element f , we need to understand the Galois structure of the corresponding extensions. We will prove

a ramified version of Lemma 5.8 of [CP14]. Following the notation of that article (which is the original notation of [Ram99]), let $K = \mathbb{Q}(Ad^0 \bar{\rho})\mathbb{Q}(\mu_p)$. For each f_i let L_i the extension of K given by its kernel and for each ϕ_j let M_j be the corresponding one. Finally let $K' = K \cdot \mathbb{Q}(Ad^0 \rho_n)$ and L_f be the extension of K given by f . Let $L = L_f \prod L_i$ and $M = \prod M_j$. We claim that $K' \cap LM = L_f$.

For this, let $\mathcal{H} = \text{Gal}(K'/K) \subseteq \text{PGL}_2(\mathcal{O}/\pi^n)$ and $\pi_1 : \text{PGL}_2(\mathcal{O}/\pi^n) \rightarrow \text{PGL}_2(\mathbb{F})$. Observe that \mathcal{H} consists on the classes of matrices in $\text{Im}(\rho_n)$ which are trivial in $\text{PGL}_2(\mathbb{F})$, i.e. $\mathcal{H} = \text{Im}(Ad^0 \rho_n) \cap \text{Ker}(\pi_1)$. Recall that our hypotheses imply $\text{PSL}_2(\mathcal{O}/\pi^n) \subseteq \text{Im}(Ad^0 \rho_n) \subseteq \text{PGL}_2(\mathcal{O}/\pi^n)$, and therefore $\text{PSL}_2(\mathcal{O}/\pi^n) \cap \text{Ker}(\pi_1) \subseteq \mathcal{H} \subseteq \text{Ker}(\pi_1)$. As $[\text{PSL}_2(\mathcal{O}/\pi^n) : \text{PGL}_2(\mathcal{O}/\pi^n)] = 2$ and $\text{Ker}(\pi_1)$ is a p group we have that $\mathcal{H} = \text{Ker}(\pi_1)$.

Recall that $\text{Gal}(F/K) \simeq (Ad^0 \bar{\rho})^r \times (Ad^0 \bar{\rho}^*)^s$ as $\mathbb{Z}[G_{\mathbb{Q}}]$ -module and by Lemma 7 of [Ram99], this is its decomposition as $\mathbb{Z}[G_{\mathbb{Q}}]$ simple modules. This implies that $K' \cap LM$ is the direct sum of the quotients of $\text{Gal}(K'/K) \simeq \mathcal{H}$ isomorphic to $Ad^0 \bar{\rho}$ or $(Ad^0 \bar{\rho})^*$. To prove that the only such quotient is $\text{Gal}(L_f/K)$ observe that any surjective morphism $\mathcal{H} \rightarrow Ad^0 \bar{\rho}$ or $(Ad^0 \bar{\rho})^*$ must contain $[\mathcal{H} : \mathcal{H}]$ inside its kernel. We will prove in Lemma 4.3 below that such commutator is equal to the subgroup of \mathcal{H} formed by the matrices congruent to the identity modulo π^2 . This finishes the proof as implies that such quotient necessarily factors through $\text{Gal}(L_f/K)$. \square

Lemma 4.3. *If $H \subseteq \text{SL}_2(\mathcal{O}/\pi^n)$ is the subgroup consisting of matrices congruent to the identity modulo p then its commutator subgroup $[H : H]$ is the subgroup H' of $\text{SL}_2(\mathcal{O}/\pi^n)$ formed by the matrices congruent to the identity modulo π^2 .*

Proof. It is easy to check that H/H' is abelian, implying that $[H : H] \subseteq H'$. For the other inclusion, observe that H' is generated by the set of elements of the form

$$\begin{pmatrix} 1 & \pi^2 x \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ \pi^2 y & 1 \end{pmatrix} \text{ and } \begin{pmatrix} 1 + \pi^2 z & 0 \\ 0 & (1 + \pi^2 z)^{-1} \end{pmatrix}$$

for $x, y, z \in \mathcal{O}$. It is easy to verify the following identities for any $a, b \in \mathcal{O}$:

- $\left[\begin{pmatrix} 1 & \pi a \\ 0 & 1 \end{pmatrix} : \begin{pmatrix} 1+\pi & 0 \\ 0 & (1+\pi)^{-1} \end{pmatrix} \right] = \begin{pmatrix} 1 - \pi^2 a(\pi+2) & 0 \\ 0 & 1 \end{pmatrix} \in [H : H].$
- $\left[\begin{pmatrix} 1 & 0 \\ \pi b & 1 \end{pmatrix} : \begin{pmatrix} 1+\pi & 0 \\ 0 & (1+\pi)^{-1} \end{pmatrix} \right] = \begin{pmatrix} 1 & 0 \\ b\pi^2 \frac{1}{(\pi+1)^2} & 1 \end{pmatrix} \in [H : H].$

This shows that the first two families of generators of H' lie inside $[H : H]$. In order to prove that $[H : H] = H'$ it only remains to check that $\begin{pmatrix} 1 + \pi^2 z & 0 \\ 0 & (1 + \pi^2 z)^{-1} \end{pmatrix} \in [H : H]$ for any $z \in \mathcal{O}$. But $\left[\begin{pmatrix} 1 & 0 \\ \pi & 1 \end{pmatrix} : \begin{pmatrix} 1 & \pi c \\ 0 & 1 \end{pmatrix} \right] = \begin{pmatrix} 1 - c\pi^2 & c^2\pi^3 \\ -c\pi^3 & c^2\pi^4 + c\pi^2 + 1 \end{pmatrix} \in [H : H]$. Multiplying this element by matrices of the form $\begin{pmatrix} 1 & 0 \\ \pi^2 x & 1 \end{pmatrix}$ we get $\begin{pmatrix} 1 - c\pi^2 & c^2\pi^3 \\ 0 & (1 - c\pi^2)^{-1} \end{pmatrix} \in [H : H]$ (as we can raise the power of π appearing in the place (2, 1) eventually making it 0 modulo π^n). The same argument applies to matrices of the form $\begin{pmatrix} 1 & \pi^2 y \\ 0 & 1 \end{pmatrix}$ we get $\begin{pmatrix} 1 - c\pi^2 & 0 \\ 0 & (1 - c\pi^2)^{-1} \end{pmatrix} \in [H : H]$. \square

Lemma 4.2 will let us prove the existence of auxilliary primes that kill global obstructions. We introduced special primes because otherwise we would have not been able to modify the behaviour of f .

Lemma 4.4. *Let ρ_n and P as before. Then there exists a finite set P' consisting of nice primes for ρ_n and eventually one special prime for f such that $\text{III}_{P \cup P'}^1(Ad^0 \bar{\rho})$ and $\text{III}_{P \cup P'}^2(Ad^0 \bar{\rho})$ are both trivial.*

Proof. If $f \notin \text{III}_P^1(Ad^0 \bar{\rho})$ this follows from taking basis $\{f_1, \dots, f_r\}$ and $\{\phi_1, \dots, \phi_s\}$ of $\text{III}_P^1(Ad^0 \bar{\rho})$ and $\text{III}_P^2(Ad^0 \bar{\rho})$ respectively and choosing, by applying Lemma 4.2, sets of nice primes q_1, \dots, q_r and q'_1, \dots, q'_s such that

- $f_i|_{G_{q_j}} = 0$ if $i \neq j$ and $f_i|_{G_{q_i}} \neq 0$.
- $\phi_i|_{G_{q'_j}} = 0$ if $i \neq j$ and $\phi_i|_{G_{q'_i}} \neq 0$.

If, otherwise, $f \in \text{III}_P^1(\text{Ad}^0 \bar{\rho})$, we do the same but taking a special prime for f instead of a nice prime. \square

From the previous lemmas, we can assume that $\text{III}_P^1(\text{Ad}^0 \bar{\rho})$ and $\text{III}_P^2(\text{Ad}^0 \bar{\rho})$ are both trivial by enlarging P if necessary. This imply, as in [KLR05], the following two key propositions.

Proposition 4.5. *Let S be a finite set of primes and $\rho_m : G_S \rightarrow \text{GL}_2(\mathcal{O}/\pi^m)$ a continuous representation such that $\text{III}_S^1(\text{Ad}^0 \bar{\rho}) = \text{III}_S^2(\text{Ad}^0 \bar{\rho}) = 0$. Then there exists a set Q of nice primes for ρ_m such that the map*

$$H^1(G_{S \cup Q}, \text{Ad}^0 \bar{\rho}) \rightarrow \bigoplus_{v \in S} H^1(G_v, \text{Ad}^0 \bar{\rho})$$

is an isomorphism.

Proof. Given the existence of auxilliary primes, this is just Lemma 8 of [KLR05]. \square

Proposition 4.6. *Let ρ_m , S and Q as in the Proposition 4.5. For each $q_i \in Q$ pick an element $h_i \in H^1(G_{q_i}, \text{Ad}^0 \bar{\rho})$. Then there is a finite set T of nice primes for ρ_m and an element*

$$g \in H^1(G_{S \cup Q \cup T}, \text{Ad}^0 \bar{\rho})$$

satisfying

- $g|_{G_v} = 0$ for $v \in S$.
- $g(\sigma_{q_i}) = h_i(\sigma_{q_i})$ for $q_i \in Q$.
- $g(\sigma_v) = 0$ for $v \in T$.

Proof. This is Corollary 11 of [KLR05], which follows from Lemmas 8 and 9 and Proposition 10 from the same work. It can be easily checked that the proofs given in that paper adapt well to our setting. \square

We are now able to state and prove the main theorem of this section. Recall from for v a nice prime there is a set C_v of deformations to characteristic 0 and a subspace $N_v \subseteq H^1(G_v, \text{Ad}^0 \bar{\rho})$ preserving its reductions.

Theorem 4.7. *Let ρ_n and P as in Theorem 4.1 and let α be an integer greater or equal than n . Pick, for each $v \in P$ a lift*

$$\rho_{v,\alpha} : G_v \rightarrow \text{GL}_2(\mathcal{O}/\pi^\alpha)$$

of $\rho_n|_{G_v}$. Then, there is a finite set of nice primes P' for ρ_n and a lift

$$\rho_\alpha : G_{P \cup P'} \rightarrow \text{GL}_2(\mathcal{O}/\pi^\alpha)$$

of ρ_n such that $\rho_\alpha|_{G_v} \simeq \rho_{v,\alpha}$ for every $v \in P$ and $\rho_\alpha|_{G_v}$ is a reduction of some member of C_v for every $v \in P'$.

Proof. We will prove the theorem by induction on α . If $\alpha = n$ the statement is trivial. Assume the theorem is true for $\alpha = m$, and apply it with the collection of local deformations given by the reductions mod π^m of the local representations $\rho_{v,m+1}$. Then, there is a lift $\rho_m : G_{P \cup P'} \rightarrow \text{GL}_2(\mathcal{O}/\pi^m)$ such that $\rho_m|_{G_v} = \rho_{v,m+1} \pmod{\pi^m}$, where P' consists on nice primes for ρ_n . We will add two sets of nice primes in order to first get a lift of ρ_m to \mathcal{O}/π^{m+1} and then locally adjust this lift. Since $\text{III}_P^2(\text{Ad}^0 \bar{\rho}) = 0$, ρ_m has no global obstructions. Also observe that ρ_m is unobstructed at the primes of P , as $\rho_m|_{G_v}$ lifts to $\rho_{v,m+1}$ and at the primes of P' too, as $\rho_m|_{G_v}$ is the reduction of some member of C_v . Therefore ρ_m is both globally and locally unobstructed implying that it lifts to some

$$\tilde{\rho}_{m+1} : G_{P \cup P'} \rightarrow \text{GL}_2(\mathcal{O}/\pi^{m+1}).$$

To complete the proof, we need to fix the local behaviour of $\tilde{\rho}_{m+1}$. We will do this in two steps. First of all, pick for each $v \in P \cup P'$ a class $u_v \in H^1(G_v, \text{Ad}^0 \bar{\rho})$ such that

- $(1 + \pi^m u_v) \tilde{\rho}_{m+1}|_{G_v} \simeq \rho_{v,m+1}$ for $v \in P$.
- $(1 + \pi^m u_v) \tilde{\rho}_{m+1}|_{G_v}$ is a reduction of a member of C_v for $v \in P'$.

Now, let Q be the set of nice primes produced by applying Proposition 4.5 to $S = P \cup P'$. As the map

$$H^1(G_{P \cup P' \cup Q}, Ad^0 \bar{\rho}) \rightarrow \bigoplus_{v \in P \cup P'} H^1(G_v, Ad^0 \bar{\rho})$$

is an isomorphism, there is a class $g_1 \in H^1(G_{P \cup P' \cup Q}, Ad^0 \bar{\rho})$ such that $g_1|_{G_v} = u_v$ for all $v \in P \cup P'$. Acting by this element on $\tilde{\rho}_{m+1}$ fixes its local shape at the places of $P \cup P'$ but may ruin it at the newly added primes of Q . We will solve this issue by adding a second set of auxilliary primes.

We pick, for each $q_i \in Q$ a class $h_i \in H^1(G_{q_i}, Ad^0 \bar{\rho})$ such that

$$(1 + \pi^m(h_i + g_1))\tilde{\rho}_{m+1}(\sigma_{q_i}) = \begin{pmatrix} q_i & 0 \\ 0 & 1 \end{pmatrix}.$$

Let T and g_2 be respectively the set of nice primes and the element of $H^1(G_{P \cup P' \cup Q \cup T}, Ad^0 \bar{\rho})$ obtained from applying Proposition 4.6 with $S = P \cup P'$ and Q and h_i as above. It is easy to check that

- $(1 + \pi^m g)\tilde{\rho}_{m+1}|_{G_v} \simeq \rho_{v,m+1}$ for $v \in P$.
- $(1 + \pi^m g)\tilde{\rho}_{m+1}|_{G_v} \in C_v$ for $v \in P' \cup Q \cup T$.

It follows that $\rho_{m+1} = (1 + \pi^m g)\tilde{\rho}_{m+1}$ satisfies what we need, completing the proof. \square

4.2. Exponent α and above. Assume we have a representation ρ_n as in Theorem 4.1 with $n \geq \alpha$ (since otherwise we apply Theorem 4.7). To ease the notation, let P denote the set $P \cup P'$ if we applied Theorem 4.7.

Recall from Sections 2 and 3 that for exponents bigger than α we have defined for each $v \in P$ a set of deformations C_v of ρ_n to characteristic 0 and a subspace $N_v \subseteq H^1(G_v, Ad^0 \bar{\rho})$ such that N_v preserve the reductions of the elements in C_v , in the sense of Proposition 2.1. They also satisfy that $\dim N_v = \dim H^1(G_v, Ad^0 \bar{\rho}) - \dim H^2(G_v, Ad^0 \bar{\rho})$ for $v \in P \setminus \{p\}$ and $\dim N_p = \dim H^1(G_p, Ad^0 \bar{\rho}) - \dim H^2(G_p, Ad^0 \bar{\rho}) + 1$.

In this setting, we can mimic the arguments of [Ram02] in our situation with some minor modifications. We start by collecting a series of results that will prove useful.

Lemma 4.8. *Let $r = \dim \text{III}_P^1((Ad^0 \bar{\rho})^*)$ and $s = \sum_{v \in P} \dim H^2(G_v, Ad^0 \bar{\rho})$. Then*

$$\dim H^1(G_P, Ad^0 \bar{\rho}) = r + s + 2.$$

Proposition 4.9. *Let $\{f, f_1, \dots, f_{r+s+1}\}$ be a basis of $H^1(G_P, Ad^0 \bar{\rho})$, where f is the element attached to $\rho_n \bmod \pi^2$. There is a set $Q = \{q_1, \dots, q_r\}$ of nice primes for ρ_n not in P such that:*

- $\text{III}_{P \cup Q}^1((Ad^0 \bar{\rho})^*) = 0$ and $\text{III}_{P \cup Q}^2(Ad^0 \bar{\rho}) = 0$.
- $f_i|_{G_{q_j}} = 0$ for $i \neq j$ and $f|_{q_j} = 0$ for all j .
- $f_i|_{G_{q_i}} \notin N_{q_i}$
- The inflation map $H^1(G_P, Ad^0 \bar{\rho}) \rightarrow H^1(G_{P \cup Q}, Ad^0 \bar{\rho})$ is an isomorphism.

Proof. Lemma 4.8 is just the Lemma before Lemma 10 in [Ram02]. The proof of Proposition 4.9 mimics that of Fact 16 of [Ram02]. Observe that, in the spirit of what is remarked in the proof of Proposition 4.2, the condition for a prime q being nice for ρ_n is not compatible with $f|_{G_q} \notin N_q$ as every subspace N_q is chosen such that $f|_{G_q} \in N_q$. In particular, f is always in the kernel of the restriction map

$$H^1(G_{P \cup Q}, Ad^0 \bar{\rho}) \rightarrow \bigoplus_{v \in P \cup Q} H^1(G_v, Ad^0 \bar{\rho})/N_v.$$

As we have checked in Proposition 4.2, conditions for the rest of the f_i 's are independent from being ρ_n -nice, so the same proof as in [Ram02] works. \square

Lemma 4.10. *Let $\langle f, f_1, \dots, f_d \rangle$ be the kernel of the map*

$$H^1(G_{P \cup Q}, Ad^0 \bar{\rho}) \rightarrow \bigoplus_{v \in P} H^1(G_v, Ad^0 \bar{\rho})/N_v.$$

Then $r \geq d$.

Proof. Follows from the formulas $\dim \oplus_{v \in P} H^1(G_v, Ad^0 \bar{\rho})/N_v = s + 1$ and $\dim H^1(G_{P \cup Q}, Ad^0 \bar{\rho}) = r + s + 2$. \square

Lemma 4.11. *There is a finite set of nice primes $\{t_{r+1}, \dots, t_d\}$ for ρ_n such that*

- $f_i|_{t_j} = 0$ if $i \neq j$ and $f_i|_{t_i} \notin N_{t_i}$.
- The restriction map

$$H^1(G_{P \cup Q \cup T}, Ad^0 \bar{\rho}) \rightarrow \bigoplus_{v \in P} H^1(G_v, Ad^0 \bar{\rho})/N_v$$

is surjective.

Proof. This is Lemma 14 of [Ram02], which relies on Proposition 10. It is easy to check that the proof given there adapts to the ramified setting, using the results we have available so far, as it does not involve picking nice primes that satisfy properties at f . \square

The set $Q \cup T$ will serve as the auxilliary set for Theorem 4.1. So far we have the following properties:

- For $1 \leq i \leq r$: $f_i|_{G_v} = 0$ for all $v \in Q \cup T$ except $q_i \in Q$ for which $f_i|_{G_{q_i}} \notin N_{q_i}$.
- For $r + 1 \leq i \leq d$: $f_i|_{G_v} = 0$ for all $v \in Q \cup T$ except $t_i \in T$ for which $f_i|_{G_{t_i}} \notin N_{t_i}$.
- The restriction $\langle f_{d+1}, \dots, f_{d+s+1} \rangle \rightarrow \oplus_{v \in P} H^1(G_v, Ad^0 \bar{\rho})/N_v$ is an isomorphism.
- $f|_{G_v} \in N_v$ for every $v \in P \cup Q \cup T$.

It easily follows from these properties that

Proposition 4.12. *The map*

$$H^1(G_{P \cup Q \cup T}, Ad^0 \bar{\rho}) \rightarrow \bigoplus_{v \in P \cup Q \cup T} H^1(G_v, Ad^0 \bar{\rho})/N_v$$

is surjective and has one dimensional kernel generated by f .

From this, we can easily deduce Theorem 4.1 in the same way as Theorem 1 is proved in [Ram02]. Moreover we have the following result

Theorem 4.13. *Let $\rho_n : G_{\mathbb{Q}} \rightarrow GL_2(\mathcal{O}/\pi^n)$ and $\rho_v : G_v \rightarrow GL_2(\mathcal{O})$ as in Theorem 4.1. Consider the collection \mathcal{L} of deformation conditions given by the pairs (C_v, N_v) for $v \in P \cup Q \cup T$. Then the deformation problem with fixed determinant and local conditions \mathcal{L} has universal deformation ring $R_u \simeq W(\mathbb{F})[[X]]$.*

Proof. Proposition 4.12 tells us that $H_{\mathcal{L}}^1(G_{P \cup Q \cup T}, Ad^0 \bar{\rho}) = \langle f \rangle$. As $\text{III}_{P \cup Q \cup T}^2(Ad^0 \bar{\rho}) = 0$, we also know that the problem is unobstructed. This proves the theorem. \square

5. MODULARITY

So far we have constructed, for a mod π^n representation ρ_n , which is nearly ordinary at p , a global deformation $\rho_u : G_{\mathbb{Q}} \rightarrow GL_2(W(\mathbb{F})[[X]])$ that lifts ρ_n and is also nearly ordinary at p . This gives a family of lifts of ρ_n to rings of dimension and characteristic 0. The purpose of this section is to prove that when ρ_n is *ordinary* at p , at least one of these lifts is modular. However, getting a lift with a previously fixed weight seems out of the scope of this work. We will prove the following.

Theorem 5.1. *Let p be a prime, \mathcal{O} the ring of integers of a finite extension K/\mathbb{Q}_p with ramification degree $e > 1$ and π its local uniformizer. Let $\rho_n : G_{\mathbb{Q}} \rightarrow GL_2(\mathcal{O}/\pi^n)$ be a continuous representation satisfying*

- ρ_n is odd.
- $\text{Im}(\rho_n)$ contains $SL_2(\mathcal{O}/p)$ if $n > e$ and $SL_2(\mathcal{O}/\pi^n)$ otherwise.
- ρ_n is ordinary and not scalar at p .

Let P be a set of primes containing the set of ramification of ρ_n , and for each $v \in P$ pick a local deformation $\rho_v : G_v \rightarrow GL_2(\mathcal{O})$ lifting $\rho_n|_{G_v}$ which is not bad. Then there exists a finite set of primes Q and a continuous representation $\rho : G_{P \cup Q} \rightarrow GL_2(\mathcal{O})$ such that

- ρ lifts ρ_n , i.e. $\rho \equiv \rho_n \pmod{\pi^n}$.

- ρ is modular.
- For every $v \in P$, $\rho|_{I_v} \simeq \rho_v|_{I_v}$ over $\mathrm{GL}_2(\mathcal{O})$.
- For every $q \in Q$, $\rho|_{I_q}$ is unipotent and $q \not\equiv \pm 1 \pmod{p}$ for all but possibly one $q \in Q$.
- ρ is ordinary at p .

Proof. Theorem 4.13 implies the existence of an universal ring $R_u \simeq \mathbb{Z}_p[[X]]$ that parametrizes nearly ordinary deformations with fixed determinant $\omega\chi^k$ and satisfying certain local conditions at the primes of $P \cup Q$.

Observe that for each morphism of local \mathbb{Z}_p -algebras $\gamma : R_u \rightarrow \mathcal{O}$ we have a nearly ordinary deformation ρ_γ with coefficients in \mathcal{O} . If we look at the local structure at p of this deformation we find that

$$\rho_\gamma|_{I_p} \simeq \begin{pmatrix} \omega_1\psi_1\chi^b & * \\ 0 & \omega_2\psi_2\chi^{k-b} \end{pmatrix}$$

for ω_1, ω_2 unramified characters, ψ_1 and ψ_2 of finite order and $b \in \mathbb{Z}_p$ (using that any p -adic character can be written as a power of the cyclotomic character times a character of finite order times an unramified character). Twisting ρ_γ by $\psi_2^{-1}\chi^{b-k}$ we get, for each $\gamma \in \mathrm{Hom}_{\mathrm{loc}-\mathbb{Z}_p}(R_u, \mathcal{O})$ an ordinary deformation of $\bar{\rho}$. Denote the representation $\psi_2^{-1}\chi^{b-k}\rho_\gamma$ by $\tilde{\rho}_\gamma$.

As we are asking $\bar{\rho}$ to be modular, the work of [DT94] ensures that R_u contains a twist of characteristic zero modular points. This is, there is a morphism $\gamma_k : R_u \rightarrow \overline{\mathbb{Q}_p}$ such that the twisted deformation $\tilde{\rho}_{\gamma_k}$ is ordinary of weight k (and therefore modular). Specifically, [DT94] guarantees that $\bar{\rho}$ has an ordinary lift of classical weight and arbitrary determinant. After twisting this lift by the corresponding power of the cyclotomic character, it lies in our family of deformations with fixed determinant. This implies that the family of representations that we have constructed is part of the Hida family of $\tilde{\rho}_{\gamma_k}$. Let \mathcal{H} be this Hida family, then there is a morphism $\Omega : \mathrm{Hom}_{\mathrm{loc}-\mathbb{Z}_p}(R_u, \mathcal{O}) \rightarrow \mathcal{H}$. As ρ_n admits different lifts to \mathcal{O} , this morphism is not constant. Recall this Hida family is equipped with his corresponding weight map $w : H \rightarrow W$.

On the other hand, we have a morphism $\theta : \mathbb{Z}_p[[X]] \rightarrow \mathcal{O}/\pi^n$ which induces ρ_n . We know that any morphism $\tilde{\theta} : \mathbb{Z}_p[[X]] \rightarrow \mathcal{O}$ that lifts θ induces a lift $\rho_{\tilde{\theta}}$ of ρ_n to \mathcal{O} . But $\mathrm{Hom}_{\mathrm{loc}-\mathbb{Z}_p}(\mathbb{Z}_p[[X]], \mathcal{O}) \simeq \mathcal{M}_{\mathcal{O}}$ (as defining such a morphism amounts to choosing an element of the maximal ideal of \mathcal{O} where to send X) and the set of morphisms $\tilde{\theta}$ that lift θ correspond to an open set $\mathcal{U} \subseteq \mathcal{M}_{\mathcal{O}}$ under this identification.

Overall, if we restrict the previously constructed morphism we have $\Omega|_{\mathcal{U}} : \mathcal{U} \rightarrow \mathcal{H}$. The image of this morphism must be an open set (as Ω is analytic), and if we compose it with the weight map we get an open set $w \circ \Omega(\mathcal{U}) \subseteq W$. This open set necessarily contains a classical point, and any pre-image of this classical point gives rise to an ordinary lift of ρ_n of integer weight which we call ρ . Finally, by the main theorem of [SW01], ρ is modular. \square

Corollary 5.2. *Let ρ_n in the same hypotheses as in Theorem 5.1. Then there exists a finite set of primes Q and a continuous representation $\rho : G_{P \cup Q} \rightarrow \mathrm{GL}_2(\mathcal{O})$ satisfying the consequences of Theorem 5.1 which also is of weight 2.*

Proof. Once we have a lift of ρ_n to characteristic 0 (which exists by Theorem 5.1) the corollary follows from the results of Section 5 of [KR14]. This gives another lift of ρ_n which may ramify at a bigger set of primes but is of weight 2. \square

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